DEFINITE INTEGRATION

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Symbol : $\displaystyle\int_{x=a}^{x=b} f(x)\,dx$ OR $\displaystyle\int_a^b f(x)\,dx$

b — upper limit.
a — lower limit

• **Fundamental Theorem of Calculus –**

Let f be a fn^n of 'x' defined on $[a,b]$ and $F(x)$ be another fn^n s.t $\frac{d(F(x))}{dx} = f(x)$ $\forall\ x \in [a,b]$, then

$$\int_a^b f(x)\,dx = \Big[F(x)+C\Big]_a^b = F(b) - F(a)$$

Q (i) $\displaystyle\int_0^1 \frac{1}{3+4x}\,dx$ (ii) $\displaystyle\int_0^{\pi/4} s^4_x\,dx$

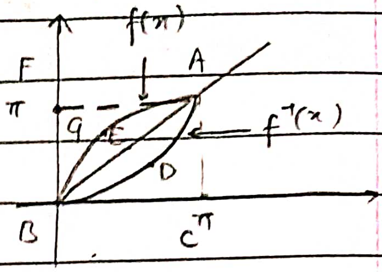
☆ (iii) If $f: R \to R$, $f(x) = s_x + x$, then find $\displaystyle\int_0^{\pi} f'(x)\,dx$

A. (i) $\left[\frac{l(4x+3)}{4}\right]_0^1 = \frac{l(7/3)}{4}$

Check (ii) $s^4_x = \frac{1}{4}(1-C_{2x})^2 = \frac{1}{4} - \frac{C_{2x}}{2} + \frac{C^2_{2x}}{4} = \frac{3}{8} - \frac{C_{2x}}{2} + \frac{C_{4x}}{8}$; $C^2_{2x} = \frac{1+C_{4x}}{2}$

$$\int_0^{\pi/4} s^4_x\,dx = \left[\frac{3x}{8} - \frac{s_{2x}}{4} + \frac{s_{4x}}{32}\right]_0^{\pi/4} = \frac{3\pi}{32} - \frac{1}{4}$$

★ (iii) We need to find [ADBC]



$$= [AEBC] - [ADB]$$

$$= [AEBC] - \big([ADBF] - [AFB]\big)$$

$$= \underbrace{[AEBC] + [AFB]}_{\pi^2} - \underbrace{[ADBF]}_{\int_0^{\pi} \sin x + x \, dx}$$

$$= \pi^2 - \left(\frac{\pi^2}{2} + 2\right)$$

$$= \pi^2/2 - 2$$

<u>Alternate Method</u> : $u = f^{-1}(x) \Rightarrow x = f(u)$

$\Rightarrow dx = f'(u)\, du$

$$\Rightarrow \int_0^{\pi} f^{-1}(x)\, dx$$

$$= \int_0^{\pi} u f'(u)\, du$$

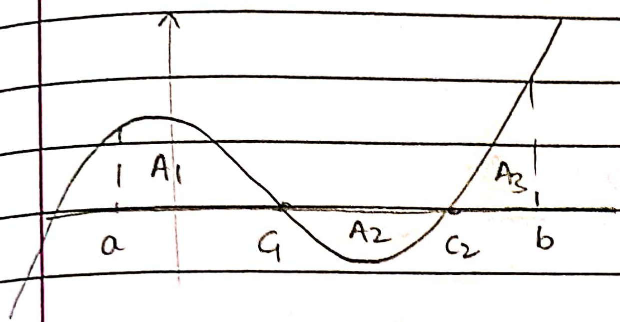
D	I
u	$f'(u)$
du	$f(u)$

$$= \big[u f(u)\big]_0^{\pi} - \int_0^{\pi} f(u)\, du$$

$$= \big[u f(u)\big]_0^{\pi} - \int_0^{\pi} \sin u + u \, du = \left[u f(u) + \cos u - \frac{u^2}{2}\right]_0^{\pi}$$

$$= \left(\pi^2 - 1 - \frac{\pi^2}{2}\right) - 1 = \frac{\pi^2}{2} - 2$$

GEOMETRICAL MEANING OF DI (SIGNED AREA)



$$\int_a^b f(x)\,dx = A_1 - A_2 + A_3$$

Area under curve

$$= \left|\int_a^{c_1} f(x)\,dx\right| + \left|\int_{c_1}^{c_2} f(x)\,dx\right| + \left|\int_{c_2}^{b} f(x)\,dx\right|$$

$$= \int_a^b |f(x)|\,dx$$

METHOD OF SUB^N

Q. $\displaystyle\int_0^{\pi/2} \frac{dx}{a^2c^2 + b^2s^2}$, $a, b > 0$

A $\displaystyle\int_0^{\pi/2} \frac{\sec^2 dx}{a^2+b^2t^2}$

$u = t$

$du = \sec^2(x)\,dx$

$\int_0^{\pi/2} \rightarrow \int_0^{\infty}$

$$\Rightarrow \int_0^{\infty} \frac{du}{b^2u^2+a^2} = \left[\frac{1}{ab}\, t^{-1}\left(\frac{bu}{a}\right)\right]_0^{\infty} = \frac{\pi}{2ab}$$

PROPERTIES OF DI

1. $$\int_a^b f(x)\,dx = \int_a^b f(u)\,du$$

2. $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

3. $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Here, c may or may not be $\in (a,b)$

Q (i) $\int_0^{\pi} |cx|\,dx$ (ii) $\int_{\pi/2}^{3\pi/2} [2sx]\,dx$

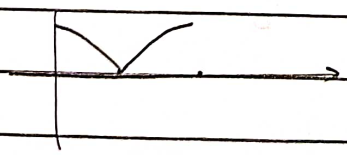
(iii) $\int_0^{2} \{x\}\,dx$ (iv) $\int_0^{1} \{\sqrt{x}\}\,dx$

(v) $\int_0^{100} [t^{-1}(x)]\,dx$ (vi) $\int_{-2}^{2} \min(\{x\}, \{-x\})\,dx$

(vii) $\int_0^{2\pi} [|s|+|c|]\,dx$ (viii) $\int_{-\pi/2}^{2\pi} [\cot^{-1}(x)]\,dx$

(ix) $\int_0^{\pi/4} [s+[c+[t+[sec]]]]\,dx$ (x) $\int_0^{2} |x^2+2x+3|\,dx$

A (i)



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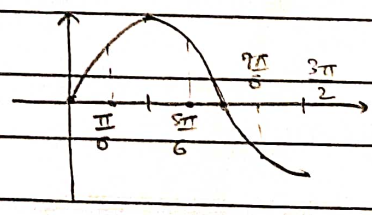
(ii)

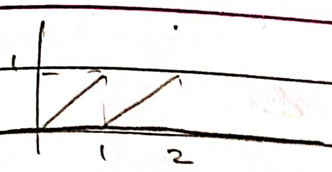


$$\left(\frac{5\pi}{6}-\frac{\pi}{2}\right)(1) + \left(\frac{7\pi}{6}-\pi\right)(-1)$$

$$+\left(\frac{3\pi}{2}-\frac{7\pi}{6}\right)(-2)$$

$$= \frac{2\pi}{6} - \frac{\pi}{6} - \frac{4\pi}{6} = -\pi/2$$

(iii)



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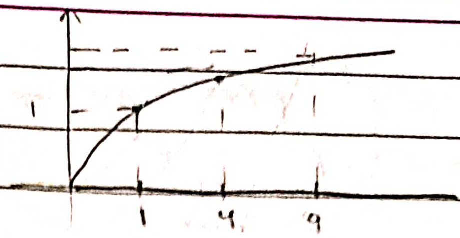
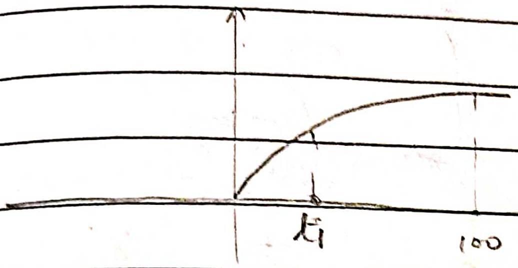
(iv)



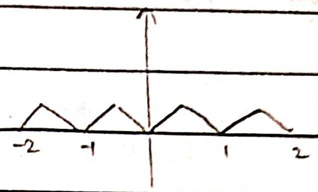
$$= \int_0^1 \sqrt{x}\,dx + \int_1^4 \sqrt{x}-1\,dx + \int_4^9 \sqrt{x}-2\,dx = 5$$

(v)



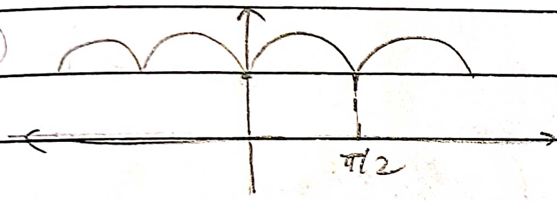
$(100-t_1)$

(vi)



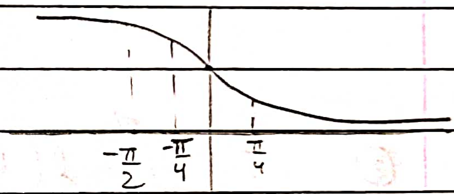
$4\left(\frac{1}{2}\right)(1)\left(\frac{1}{2}\right) = \underline{1}$

(vii)



2π

(viii)



$\left(\frac{\pi}{4}\right)(2) + \left(\frac{\pi}{2}\right)(1) = \pi$

(ix) $[\sec(x)] = 1 \quad x \in [0, \pi/4]$

$\in[0,1]$

$\Rightarrow [s + [c + [t] + 1]]$

$\Rightarrow [s + [c] + 1] = [s] + 1$

$= 1$

$\int_0^{\pi/4} 1\,dx = \underline{\pi/4}$

(x) $\int_0^2 x^2+2x+3\,dx$

$= \left(\frac{x^3}{3} + x^2 + 3x\right)_0^2$

$= \frac{8}{3} + 4 + 6 = \underline{38/3}$

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4. $$\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$$

eg ① $$I = \int_0^{\pi/2} \frac{dx}{1+\sqrt{tx}} = \int_0^{\pi/2} \frac{dx}{1+\sqrt{t_{(\frac{\pi}{2}-x)}}}$$

$$= \int_0^{\pi/2} \frac{dx}{1+\frac{1}{\sqrt{tx}}} = \int_0^{\pi/2} \frac{\sqrt{tx}\, dx}{1+\sqrt{tx}}$$

$$2I = \int_0^{\pi/4} \frac{1+\sqrt{tx}\; dx}{1+\sqrt{tx}} = \int_0^{\pi/4} dx = \pi/4 \Rightarrow \underline{I = \pi/8}$$

② $$I = \int_0^{\pi/4} l(1+t)\, dx = \int_0^{\pi/4} l(1+\underbrace{t_{(\frac{\pi}{4}-x)}}_{\frac{1-t}{1+t}})\, dx$$

$$= \int_0^{\pi/4} l(2) - l(1+t)\, dx$$

$$2I = \int_0^{\pi/4} l(1+t) + l(2) - l(1+t)\, dx = \int_0^{\pi/4} l(2)\, dx$$

$$= \pi l(2)/4 \rightarrow \underline{I = \pi l(2)/8}$$

5. $$\int_0^{2a} f(x)\, dx = \int_0^{a} f(x) + f(2a-x)\, dx$$

Proof $$\int_0^a f(x)\, dx + \underbrace{\int_a^{2a} f(x)\, dx}_{\int_a^0 f(2a-x)\,(-dx)}$$

$x = 2a - x$

$dx = -dx$

$$= \int_0^a f(x) + f(2a-x)\, dx$$

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Special Case

$$\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx \quad , \quad \text{if } f(x) = f(a-x)$$

$$= 0 \quad , \quad \text{if } f(x) = -f(a-x)$$

eg - $\displaystyle\int_0^{2\pi} s^{100} c^{99}\,dx = \int_0^{\pi} s^{100} c^{99} + s^{100}_{(2\pi-x)} c^{99}_{(2\pi-x)}\,dx$

$$= 2\int_0^{\pi} s^{100} c^{99}\,dx$$

$$= 2\int_0^{\pi/2} s^{100} c^{99} + s^{100}(-c^{99})\,dx = \underline{0}$$

6. $\displaystyle\int_{-a}^{a} f(x)\,dx = \int_0^a f(x) + f(-x)\,dx$

$$= \begin{cases} 2\int_0^a f(x)\,dx \quad , & \text{if } f(x) \text{ is even} \\ 0 \quad , & \text{if } f(x) \text{ is odd} \end{cases}$$

Q (i) $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{x^9 - 3x^5 + 7x^3 - x + 1}{c^2}\,dx$

(ii) If $f(x)$ is odd, then P.T

$$\int_{-a}^{a} \frac{f(s)}{f(s^2) + f(c)}\,dx = 0$$

(iii) $\displaystyle\int_{-2}^{0} \left(x^3 + 3x^2 + 3x + 3 + (x+1)\, c_{(x+1)}\right) dx$

(iv) $\displaystyle\int_{-1}^{1} \frac{x}{\sqrt{1-x^2}} \quad s^{-1}\left(2x\sqrt{1-x^2}\right) \quad dx$

(v) $\displaystyle\int_{-1/2}^{1/2} C_x \quad l\left(\frac{1+x}{1-x}\right) \, dx$

$\displaystyle\int_{0}^{1/2} C_x \quad l\left(\frac{1+x}{1-x}\right) \, dx$

A. (i) $\displaystyle\int_0^{\pi/4} 2\sec^2(x)\, dx = 2\left(tx\right)_0^{\pi/4} = \underline{2}$

(ii) $\displaystyle\int_0^{2} \frac{f(s)}{f(s^2)+f(c)} + \frac{\overset{-f(s)}{\cancel{f(-s)}}}{f(s^2)+f(c)} \quad dx = \underline{0}$

(iii) $\displaystyle\int_{-2}^{0} (x+1)^3 + (x+1)\, c_{(x+1)} + 2 \quad dx$ $\qquad u = x+1$

$du = dx$

$\Rightarrow \displaystyle\int_{-1}^{1} u^3 + uc_u + 2 \quad du = \int_0^1 4 \quad dx = \underline{4}$

(iv) $\displaystyle 2\int_0^1 \frac{x}{\sqrt{1-x^2}} \quad s^{-1}\left(2x\sqrt{1-x^2}\right) \quad dx$ $\qquad u = \sqrt{1-x^2}$

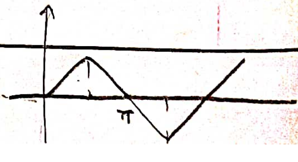
$du = \dfrac{-x}{\sqrt{1-x^2}}\, dx$

$= 2\displaystyle\int_1^{0} s^{-1}\left(2u\sqrt{1-u^2}\right)(-du)$

$= 2\displaystyle\int_0^1 s^{-1}\left(2u\sqrt{1-u^2}\right)\, du$ $\qquad v = su \qquad u \in [0,1]$

$dv = c_u\, du \qquad 2u \in [0,2]$

$= 2\displaystyle\int_0^{s_1} \frac{s^{-1}(s_{2u})\, dv}{\sqrt{1-v^2}} = 4\int_0^{s_1}$



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$$= 2\left[\int_0^{\frac{1}{\sqrt{2}}} \frac{2\overbrace{u}^{\sin^{-1}v}\,dv}{\sqrt{1-v^2}} + \int_{\frac{1}{\sqrt{2}}}^{1} \frac{\pi - 2\overbrace{u}^{\sin^{-1}v}\,dv}{\sqrt{1-v^2}}\right]$$

$$= 2\left[\sin^{-1}v^2\right]_0^{\frac{1}{\sqrt{2}}} - 2\left[\sin^{-1}v^2\right]_{\frac{1}{\sqrt{2}}}^{1} + \left[\pi \sin^{-1}(v)\right]_{1/\sqrt{2}}^{1}$$

$$= -2 + \pi\left(1 - \frac{\pi}{4}\right)$$

(v) $N = 0$ & $D \neq 0 \Rightarrow \underline{0}$

7. $$\int_a^b f(x)\,dx = (b-a)\int_0^1 f((b-a)x + a)\,dx$$

Proof $x = bu + (1-u)a$ $\Rightarrow (b-a)\int_0^1 f((b-a)u + a)\,du$

$\Rightarrow u = \frac{(x-a)}{(b-a)} \Rightarrow du = \frac{dx}{(b-a)}$

$\int_a^b \longrightarrow \int_0^1$

General: $$\int_a^b f(x)\,dx = \frac{(b-a)}{(d-c)}\int_c^d f\left(\left(\frac{b-a}{d-c}\right)x + \frac{ad-bc}{d-c}\right)dx$$

Proof: $(x-a)\left(\frac{d-c}{b-a}\right) = (u-c) \Rightarrow x = \left(\frac{b-a}{d-c}\right)(u-c) + a$

$\Rightarrow dx = \left(\frac{b-a}{d-c}\right) dt$

$\int_a^b \longrightarrow \int_c^d$

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Q (i) $\displaystyle\int_{-4}^{5} e^{(x+5)^2}\,dx + 3\int_{1/3}^{2/3} e^{9\left(x-\frac{2}{3}\right)^2}\,dx$

(ii) If $\displaystyle\int_0^1 \frac{s\,t}{1+t}\,dt = \alpha$, then find

$\displaystyle\int_{4\pi-2}^{4\pi} \frac{s(t/2)}{4\pi+2-t}\,dt$ in terms of α.

A. (i) $u = x+5$ $\Rightarrow \displaystyle\int_0^1 e^{u^2}\,du + \int_{-1}^{0} e^{v^2}\,dv$

$du = dx$

$\displaystyle\int_{-4}^{-5} \rightarrow \int_1^0$ $\Rightarrow \displaystyle\int_0^1 e^{u^2}\,du - \int_0^1 e^{v^2}\,dv$

$v = 3x-2$ $= \underline{0}$

$dv = 3dx$

$\displaystyle\int_{1/3}^{2/3} \rightarrow \int_{-1}^{0}$

(ii) $u = 4\pi - t$ $\Rightarrow \displaystyle\int_2^0 \frac{s\left(2\pi-\frac{u}{2}\right)}{4\pi+2-4\pi+u}(-du) = \int_0^2 \frac{-s(u/2)}{u+2}\,du$

$du = -dt$

$\displaystyle\int_{4\pi-2}^{4\pi} \rightarrow \int_2^0$ $\displaystyle = (2-0)\int_0^1 \frac{-s\left(\frac{(2-0)u+0}{2}\right)}{(2-0)u+0+2}\,du$

$\displaystyle = -\int_0^1 \frac{s\,u}{1+u}\,du = \underline{-\alpha}$

8. If $f(x)$ is a periodic f^n with period T, then

(I) $$\int_a^{a+nT} f(x)\,dx = \int_0^{nT} f(x)\,dx = n\int_0^{T} f(x)\,dx, \quad n \in \mathbb{Z}$$

(II) $$\int_{mT}^{nT} f(x)\,dx = (n-m)\int_0^{T} f(x)\,dx, \quad m,n \in \mathbb{Z}$$

(III) $$\int_{a+nT}^{b+nT} f(x)\,dx = \int_a^{b} f(x)\,dx$$

Q (i) $\int_0^{4\pi} |cx|\,dx$ (ii) $\int_0^{25} e^{\{x\}}\,dx$

(iii) Let $f(x) = \min(\{x+1\}, \{x-1\})$, then find $\int_{-5}^{5} f(x)\,dx$

(iv) P.T $\int_0^{n\pi+v} |sx|\,dx = (2n+1) - c_v$, $n \in \mathbb{Z}$, $v \in [0,\pi)$

(v) $\int_0^{2n\pi} [s+c]\,dx$

A. (i) $4\int_0^{\pi} |cx|\,dx$

$\Rightarrow 4(2) = \underline{8}$

(ii) $25\int_0^{1} e^{\{x\}}\,dx = 25\int_0^{1} e^{x}\,dx$

$= \underline{25(e-1)}$

(iii) $f(x) = \{x\} \Rightarrow \int_{-5}^{5} f(x)\,dx = \int_{-5}^{5} \{x\}\,dx = (10)\left(\frac{1}{2}\right) = 5$

(iv) $\underbrace{\int_0^{n\pi} |\sin x|\,dx}_{2n} + \int_{n\pi}^{n\pi+v} |\sin x|\,dx = 2n + \int_0^{v} \underbrace{|\sin x|}_{\sin x}\,dx$

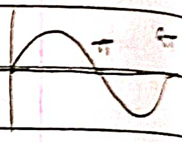
$= 2n + [-\cos x]_0^{v}$

$= (2n+1) - \cos v$

(v) $\int_0^{2n\pi} \left[\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\right] dx = \int_{\pi/4}^{2n\pi+\frac{\pi}{4}} \left[\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\right] dx = \int_0^{2n\pi} [\sqrt{2}\sin x]\,dx$

$= n\int_0^{2\pi} [\sin x]\,dx = n(-1)(2\pi-\pi)$

$= -n\pi$



9. Lebnitz Rule

(I) Let $F(x) = \int_{L(x)}^{U(x)} f(t)\,dt$, then

$$F'(x) = f(U(x))\,U'(x) - f(L(x))\,L'(x)$$

eg − $(f(x))^2 = \int_0^x f(t)\cdot \frac{2\sec^2(t)}{4+\tan t}\,dt$

and $f(0)=0$, then find $f(\pi/4)$

A. $2\cancel{f(x)}\,f'(x) = \cancel{f(x)}\,\frac{2\sec^2(x)}{4+\tan x}$

$\Rightarrow f'(x) = \frac{\sec^2(x)}{4+\tan x} \Rightarrow f(x) = \ell(4+\tan x)+C$

$f(0) = \ell(4)+C$

$\Rightarrow C = -2\ell(2)$

$\Rightarrow f\left(\frac{\pi}{4}\right) = \ell(5) - 2\ell(2) = \ell(5/4)$

* Out of syllabus

(II) If $F(t) = \int_a^b f(x,t)\,dx$, then

$$\boxed{\frac{dF}{dt} = \int_a^b \frac{\partial f(x,t)}{\partial t}\,dx}$$

eg Evaluate $\int_0^1 \frac{x^b - 1}{l(x)}\,dx$ $(b \geq 0)$

A. Let $F(b) = \int_0^1 \frac{x^b - 1}{l(x)}\,dx$

$$\Rightarrow \frac{dF(b)}{db} = \int_0^1 \frac{x^b \cancel{l(x)}}{\cancel{l(x)}}\,dx = \left(\frac{x^{(b+1)}}{b+1}\right)_0^1 = \frac{1}{(b+1)}$$

$$F(b) = l(b+1) + C.$$

$$F(0) = \int_0^1 0\,dx = 0 = l(1) + C \Rightarrow \underline{C = 0}$$

$$\Rightarrow \underline{F(b) = l(b+1)}$$

☆ Q $f(x) = x^2 + \int_0^x e^{-t} f(x-t)\, dt$

A. $f(x) = x^2 + \int_0^x e^{-(x-t)} f(x-x+t)\, dt$

$= x^2 + \int_0^x e^{-x} e^{t} f(t)\, dt$

$f'(x) = 2x + e^{-x}\left[e^{x} f(x) - \underbrace{\int_0^x e^{-t} f(t)\, dt}_{e^{x}(f(x)-x^2)}\right]$

$= 2x + f(x) - f(x) + x^2$

$= x^2 + 2x$

$f(x) = \frac{x^3}{3} + x^2 + c$ $\qquad f(0) = 0^2 + \int_0^0 e^{-t} f(x-t)\, dt$

$\qquad\qquad\qquad\qquad\qquad\quad \underline{= 0}$

$= \frac{x^3}{3} + x^2$ $\qquad \Rightarrow \underline{c = 0}$

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INEQUALITIES

(I) For a fn' $f(x)$ cont. on $[a,b]$ and are abee to find 2 fn's $f_1(x)$ & $f_2(x)$ s.t

$$f_1(x) \leq f(x) \leq f_2(x) \quad \forall \quad x \in [a,b]$$

then

$$\boxed{\int_a^b f_1(x)\, dx \leq \int_a^b f(x)\, dx \leq \int_a^b f_2(x)\, dx}$$

Special case : If $f(x) \geqslant g(x)$ on $[a,b]$

$$\Rightarrow \int_a^b f(x)\, dx \geqslant \int_a^b g(x)\, dx$$

NOTE: if $f(x) \geqslant 0$, (on $[a,b]$) then $\int_a^b f(x)\, dx \geqslant 0$

Q. (i.) P.T $\frac{\pi}{6} \le \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \le \frac{\pi}{4\sqrt{2}}$

(ii) Let $I_n = \int_0^{\pi/4} t_x^n \, dx \qquad (n>1,\ n\in \mathbb{Z})$

Then show that

(ii.i) $I_n + I_{(n-2)} = \frac{1}{(n-1)}$

(ii.ii) $\frac{1}{2(n+1)} < I_n < \frac{1}{2(n-1)}$

A(i) $\frac{1}{\sqrt{4-x^2}} \le \frac{1}{\sqrt{4-x^2-x^3}} \le \frac{1}{\sqrt{4-2x^2}} \qquad \left\{ \begin{array}{l} x^3 < x^2 \\ \forall\ x\in(0,1) \end{array} \right\}$

$$\Rightarrow \int_0^1 \frac{dx}{\sqrt{4-x^2}} \le \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \le \int_0^1 \frac{dx}{\sqrt{4-2x^2}}$$

$$\Rightarrow \frac{\pi}{6} \le \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \le \frac{\pi}{4\sqrt{2}}$$

(ii) (ii.i) $I_n + I_{(n-2)} = \int_0^{\pi/4} t^n + t^{(n-2)} \, dx = \int_0^{\pi/4} t^{(n-2)} \sec^2(x) \, dx$

$$= \left[\frac{t^{(n-1)}}{n-1}\right]_0^{\pi/4} = \frac{1}{(n-1)}$$

(ii.ii) $t^{(n+2)} < t^n < t^{(n-2)}$

$\Rightarrow I_{(n+2)} < I_n < I_{(n-2)}$

$\Rightarrow I_n + I_{(n+2)} < 2I_n < I_{(n-2)} + I_n \Rightarrow \frac{1}{2(n+1)} < I_n < \frac{1}{2(n-1)}$

Q. If $f(x)$ is a cont. fn^n s.t $f(x) \geq 0 \ \forall \ x \in [0,10]$ and $\int_4^8 f(x)\,dx = 0$, find $f(6)$.

A. $f(x) \geq 0 \Rightarrow \int_4^8 f(x) \geq 0 \Rightarrow f(x) = 0 \ \forall \ x \in [4,8]$
$\Rightarrow \underline{f(6) = 0}$

(II) If 'm' & 'M' are the min. & max. values of a fn^n $f(x)$ in the interval $[a,b]$, then

$$\boxed{m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)}$$

Q (i) P.T $1 \leq \int_0^1 e^{x^2}\,dx \leq e$

(ii) $\boxed{\int_{10}^{19} \frac{\sin x}{1+x^8}\,dx} < 9 \times 10^{-8}$

A. (i) $m = 1$, $M = e$ $\Rightarrow 1(1-0) \leq \int_0^1 e^{x^2}\,dx \leq e(1-0)$

(ii) $\frac{\sin x}{1+x^8} < \frac{1}{1+x^8} < \frac{1}{x^8} < \frac{1}{10^8}$

$\Rightarrow \int_{10}^{19} \frac{\sin x}{1+x^8}\,dx < \int_{10}^{19} \frac{dx}{10^8} = \underline{9 \times 10^{-8}}$

$$-\frac{1}{10^8} < -\frac{1}{x^8} < -\frac{1}{1+x^8} < \frac{\sin x}{1+x^8}$$

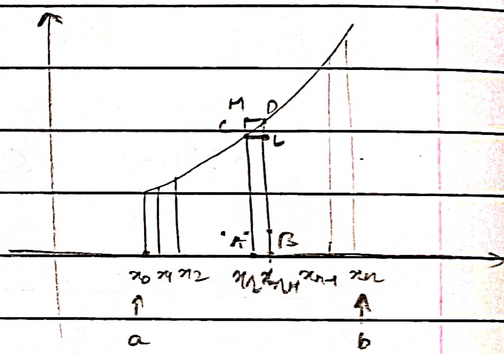
$$\Rightarrow \int_{10}^{19} -\frac{1}{10^8}\,dx < \int_0^1 \frac{\sin x}{1+x^8}$$

$$\text{LHS} = -9\times 10^{-8}$$

$$\therefore \left| \int_0^1 \frac{\sin x}{1+x^8}\,dx \right| < 9\times 10^{-8}$$

DEFINITE INTEGRAL AS LIMIT OF SUM

(Integration using First Principle)



Divide the interval $[a, b]$ into "n" equal sub-intervals. $[x_0, x_1]$, $[x_1, x_2]$ $\ldots$ $[x_{(r-1)}, x_r]$ $\ldots$ $[x_{n-1}, x_n]$

where $x_0 = a$, $x_1 = a+h$, $x_2 = a+2h$ $\ldots$, $x_r = a+rh$, $x_n = b = a+nh$

i.e $n = \frac{(b-a)}{h}$. Clearly, as $n\to\infty$, $h\to 0$

Clearly, $ar(\square ABLC) < ar(ABCDA) < ar(\square ABDM) \to (i)$

As $x_r - x_{(r-1)} \to 0$ i.e $h\to 0$, all the three areas become nearly equal.

Let us define,

$s_n = \sum$ Area of all lower rectangles $= h \sum_{r=0}^{(n-1)} f(x_r)$

$S_n = \sum$ Area of all upper rectangles $= h \sum_{r=1}^{n} f(x_r)$

(i) $\Rightarrow$ $s_n < \begin{pmatrix} \text{Area of region} \\ \text{PRSQP} \end{pmatrix} < S_n$

Hence, $\lim_{n\to\infty} S_n = \lim_{n\to\infty} s_n = \begin{pmatrix} \text{Area of} \\ \text{region PRSQP} \end{pmatrix} = \int_a^b f(x)\,dx$

OR $\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{r=0}^{(n-1)} h f(a+rh) = \lim_{n\to\infty} \sum_{r=1}^{n} h \cdot f(a+rh)}$

- Steps to express infinite series as a definite integral :-

(1) Express the given series as $\lim_{n\to\infty} \sum_{r=r_1}^{r_2} \left(\frac{1}{n}\right) f\left(\frac{r}{n}\right)$

(2) Replace $\left(\frac{r}{n}\right) \to x$, $\frac{1}{n} \to dx$, $\lim_{n\to\infty} \sum \to \int$

Lower limit : $\lim_{n\to\infty} \left(\frac{r_1}{n}\right)$

Upper limit : $\lim_{n\to\infty} \left(\frac{r_2}{n}\right)$

Q. $\lim_{n\to\infty} n\left[\frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+4)} + \cdots + \frac{1}{6n^2}\right]$

A. $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{n}{(n+r)(n+2r)} = \frac{2(n+r)-(n+2r)}{(n+r)(n+2r)}$

$$= \left(\frac{1}{n}\right)\left[\left(\frac{2}{1+2\left(\frac{r}{n}\right)}\right) - \left(\frac{1}{1+\left(\frac{r}{n}\right)}\right)\right]$$

$$\downarrow$$

$$\int_0^1 \frac{2}{1+2x} - \frac{1}{1+x}\,dx = \left[\ell\left(\frac{1+2x}{1+x}\right)\right]_0^1 = \underline{\ell(3/2)}$$

Q. (i) $\lim_{n\to\infty}\left(\frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n-1}{n^2}\right)$

(ii) $\lim_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right)$

(iii) $\lim_{n\to\infty}\left(\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \cdots + \frac{n}{2n^2}\right)$

(iv) $\lim_{n\to\infty}\frac{(1^p + 2^p + \cdots + n^p)}{n^{(p+1)}}$

A. (i) $\lim_{n\to\infty}\sum_{r=1}^{(n-1)} \frac{r}{n^2} = \left(\frac{1}{n}\right)\left(\frac{r}{n}\right)$

$$\downarrow$$

$$\int_0^1 x\,dx = \underline{1/2}$$

(ii) $\underset{n\to\infty}{lt} \sum_{r=1}^{n} \frac{1}{n+r} = \left(\frac{1}{n}\right)\left(\frac{1}{1+\frac{r}{n}}\right)$

$\downarrow$

$\int_0^1 \frac{1}{1+x}\,dx \Rightarrow \left[l(1+x)\right]_0^1 = \underline{l(2)}$

(iii) $\underset{n\to\infty}{lt} \sum_{r=1}^{n} \frac{n}{n^2+r^2} = \left(\frac{1}{n}\right)\left(\frac{1}{1+\left(\frac{r}{n}\right)^2}\right)$

$\downarrow$

$\int_0^1 \frac{dx}{1+x^2} = \left[t^{-1}(x)\right]_0^1 = \underline{\pi/4}$

(iv) $\underset{n\to\infty}{lt} \sum_{r=1}^{n} \frac{r^p}{n^{(p+1)}} = \left(\frac{1}{n}\right)\left(\frac{r}{n}\right)^p$

$\downarrow$

$\int_0^1 x^p\,dx = \left[\frac{x^{(p+1)}}{(p+1)}\right]_0^1 = \underline{1/(p+1)}$

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☆ Q. $f(x) - \lambda \int_0^{\pi/2} \sin x \cos t \, f(t)\, dt = \sin x$.

If $f(x) = 2$ has at least one real rt and $\lambda \in [a, b]$, find $a+b$

A. $f'(x) - \cos x \, \lambda \int_0^{\pi/2} \cos t \, f(t)\, dt = \cos x$

$\Rightarrow \lambda \int_0^{\pi/2} \cos t \, f(t)\, dt = \underline{\dfrac{f'(x) - \cos x}{\cos x}}$

Substituting in original eqn

$\Rightarrow f(x) - \sin x \left(\dfrac{f'(x) - \cos x}{\cos x} \right) = \sin x \Rightarrow \dfrac{f'(x)}{f(x)} = \cot(x)$

$\Rightarrow \ell(f(x)) = \ell(\sin(x)) + C$

$\Rightarrow \underline{f(x) = C \sin x}$

$\Rightarrow \cancel{C \sin x} - \lambda \sin x \int_0^{\pi/2} C \sin t \cos t \, dt = \cancel{\sin x}$

$\Rightarrow C = 1 + \dfrac{\lambda C}{4} [\cos 2t]_{\pi/2}^{0} \Rightarrow C = 1 + \dfrac{\lambda C}{2}$

$\Rightarrow \underline{C = \left(\dfrac{2}{2 - \lambda} \right)}$

$\Rightarrow \underline{f(x) = \left(\dfrac{2}{2 - \lambda} \right) \sin x}$

For $f(x) = 2 \Rightarrow \underbrace{\sin x}_{\in [-1, 1]} = 2 - \lambda \Rightarrow \boxed{\lambda \in [1, 3]}$

$\Downarrow$

$\underline{a + b = 4}$

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Q (i) $\int e^{(e^x+e^{-x})}\left(e^{2x}+2e^x-e^{-x}-1\right) = g(x)\cdot e^{(e^x+e^{-x})}$

Find $g(0)$

(ii) $\int e^{\sec(x)}\left(\sec(x)\tan(x)f(x)+\sec(x)\tan(x)\sec^2(x)\right) = e^{\sec(x)}f(x)+C$

Find $f(x)$

(iii) $f(x) = \int \left(\frac{1}{1-x^2}\right)\left(s^{-1}\left(\frac{2x}{1+x^2}\right)+t^{-1}\left(\frac{2x}{1-x^2}\right)\right)dx \quad , \quad x>1$

Find $f(x)$

☆ (iv) $\int \sqrt{x-\pi}\,\left(s^{-1}(l(x))+c^{-1}(l(x))\right)dx$

A. (i) $e^{(e^x+e^{-x})}\left(g'(x)+g(x)(e^x+e^{-x})\right) = e^{(e^x+e^{-x})}\left(e^{2x}+2e^x-e^{-x}-1\right)$

$\Rightarrow g'(x)+g(x)(e^x+e^{-x}) = e^x+(e^x+1)(e^x+e^{-x})$

$\Rightarrow g(x) = e^x+1 \quad \Rightarrow \underline{g(0)=2}$

(ii) $e^{\sec(x)}\left[f'(x)+\sec(x)\tan f(x)\right] = e^{\sec(x)}\left(\sec(x)\tan x\sec^2+\sec\tan f(x)\right)$

$\Rightarrow f'(x) = \sec^3 t \quad \Rightarrow \underline{f(x) = \frac{\sec^3(x)}{3}}$

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(iii) $x = \tan\theta$ $\Rightarrow \int \left(\frac{\sec^2}{1-t^8}\right) \left(\overbrace{\sin^{-1}(\sin 2\theta)}^{\pi-2\theta} + \overbrace{\tan^{-1}(\tan 2\theta)}^{2\theta-\pi}\right) d\theta$

$dx = \sec^2 d\theta$

$\Rightarrow f(x) = \int 0 \, d\theta \Rightarrow \underline{f(x) = C}$

☆ (iv) Fxn is <u>not</u> defined since, $\sqrt{x-\pi} \Rightarrow x \geq \pi$

$\Downarrow$ & $\sin^{-1}(\ell(x)) + \cos^{-1}(\ell(x)) \Rightarrow x \in [e^{-1}, e]$

Antiderivative does $\Downarrow$

<u>not</u> exist ! $\underline{x \in \emptyset}$

Q ☆ (v) $\int e^{(x\sin x + \cos x)} \left(\frac{x^4 \cos^3 x - x\sin x + \cos x}{x^2\cos^2 x}\right) dx$

☆ (vi) Let $f(x,n) = \int \frac{x^2 + n(n-1)}{(x\sin x + n\cos x)^2} dx$ & $f(0,n) = 0$

If $f\left(\frac{\pi}{6}, 2\right) = \frac{\lambda(4\lambda - \pi)}{4\lambda^3 + \pi}$, find $(1+\lambda+\lambda^2)(1-\lambda+\lambda^2)$

(vii) Let $F(x) = \int \frac{(1+x)\left((1-x+x^2)(1+x+x^2)+x^2\right)}{1+2x+3x^2+4x^3+3x^4+2x^5+x^6} dx$

Find $|F(99) - F(3)|$

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A ☆ (v) $\int e^{(xs+c)} \left(x^2 c - \frac{xs - c}{x^2 c^2} \right) dx$

$\Rightarrow \int x \; e^{(xs+c)} \; xc \; dx - \int e^{xs+c} \; \frac{d}{dx}\left(\frac{1}{xc}\right)$

D	I	D	I
x	$e^{(xs+c)} \; xc \; dx$	e^{xs+c}	$\frac{d}{dx}\left(\frac{1}{xc}\right)$
1	$e^{(xs+c)}$	$e^{(xs+c)}(xc)$	$\frac{1}{xc}$

$\Rightarrow x e^{(xs+c)} - \cancel{\int e^{(xs+c)} dx} - e^{(xs+c)} + \cancel{\int e^{(xs+c)} dx}$

$= (x-1) e^{(xs+c)}$

(vii) $\int \frac{(x^4 + x^2 + 1) + x^2}{x^3 \left(\left(\frac{x+1}{x}\right)^3 + 2\left(\frac{x+1}{x}\right)^2 \right)} dx = \int \frac{(x^2+1)^2}{(x^2+1)^2 (x+1)} dx = \int \frac{dx}{(x+1)}$

$= \underline{l|1+x|}$

$[F(99) - F(3)] = [l(25)] = [2l(10) - 2l(2)]$

$= [2(2.3) - 2(0.69)] = 3$

☆ (vi) $\int \frac{\frac{x^{(2n-2)}}{x^{(2n-2)}} \; x^2 + n(n-1)}{(xs + nc)^2} dx$

$= \int \frac{x^{2n} + x^{(2n-2)} n(n-1)}{(sx^n + nx^{(n-1)}c)^2} dx = \int \frac{-x^n}{c} \; d\left(\frac{1}{x^n s + nx^{(n-1)}c} \right)$

$(x^n s + nx^{(n-1)} c)'$

$= x^n c + \cancel{nx^{(n-1)}s} = \cancel{nx^{(n-1)}s}$

$+ n(n-1) x^{(n-2)} c$

$= c(x^n + x^{(n-2)} n(n-1))$

D	I
$\frac{-x^n}{c}$	$d\left(\frac{1}{x^n s + nx^{(n-1)} c}\right)$
$-\frac{(ncx^{(n-1)} + x^n s)}{c^2}$	$\frac{1}{x^n s + nc x^{(n-1)}}$

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$$\Rightarrow \frac{-x^n}{c(x^n s + nc\,x^{(n-1)})} + \int \sec^2(x)\,dx$$

$$\Rightarrow t - \frac{x^n}{c(x^n s + nc\,x^{(n-1)})} = t - \frac{1}{c\left(s+\frac{nc}{x}\right)} + C$$

$$f\left(\frac{\pi}{6}, 2\right) = \frac{1}{\sqrt{3}} - \frac{1}{\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2} + \frac{2(6)}{2\pi}\right)}$$

$f(0,n) = 0$

$\Rightarrow \boxed{C = 0}$

$$= \frac{1}{\sqrt{3}} - \frac{1}{\frac{\sqrt{3}}{4} + \frac{3}{\pi}}$$

?

$$= \frac{1}{\sqrt{3}} - \frac{4\pi}{\sqrt{3}\pi + 12} = \frac{\sqrt{3}\pi + 12 - 4\sqrt{3}\pi}{3\pi + 12\sqrt{3}} = \frac{4 - \sqrt{3}\pi}{4\sqrt{3} + \pi}$$

Q (viii) $$\int \frac{\left(1-\frac{1}{\sqrt{3}}\right)(c-s)}{\left(1+\frac{2sc}{\sqrt{3}}\right)}\,dx = \frac{1}{\alpha}\,\ell\left|\frac{t\left(\frac{x}{2}+\frac{\pi}{\beta}\right)}{t\left(\frac{x}{2}+\frac{\pi}{\gamma}\right)}\right| + C,$$

where $\alpha, \beta, \gamma \in \mathbb{Z}$.

Find $\alpha\beta\gamma$

A. $u = s+c$

$du = (s-c)\,dx$

$$\int \frac{\left(1-\frac{1}{\sqrt{3}}\right)du}{1+\frac{2}{\sqrt{3}}\left(\frac{u^2-1}{2}\right)} = \frac{\left(1-\frac{1}{\sqrt{3}}\right)}{\left(\frac{2}{\sqrt{3}}\right)}\int \frac{du}{\left(\frac{\sqrt{3}}{2}+u^2-1\right)}$$

$u = 1 + 2sc$

$\Rightarrow sc = \left(\frac{u-1}{2}\right)$

$$= \left(\frac{\sqrt{3}-1}{2}\right)\int \frac{du}{u^2 - \left(\frac{2-\sqrt{3}}{2}\right)}$$

$$= \left(\frac{\sqrt{3}-1}{2}\right)\left(\frac{1}{2}\right)\frac{1}{\sqrt{\frac{2-\sqrt{3}}{2}}}\,\ell\left|\frac{u-\sqrt{\frac{2-\sqrt{3}}{2}}}{u+\sqrt{\frac{2-\sqrt{3}}{2}}}\right|$$

$$= \frac{1}{2}\,\ell\left|\frac{s+c-\left(\frac{\sqrt{3}-1}{2}\right)}{s+c+\left(\frac{\sqrt{3}-1}{2}\right)}\right| = \frac{1}{2}\,\ell\left|\frac{s\left(x+\frac{\pi}{4}\right)-s\left(\frac{\pi}{12}\right)}{s\left(x+\frac{\pi}{4}\right)+s\left(\frac{\pi}{12}\right)}\right|$$

$$= \frac{1}{2}\,\ell\left|\frac{t\left(\frac{x}{2}+\frac{\pi}{12}\right)}{t\left(\frac{x}{2}+\frac{\pi}{6}\right)}\right| \Rightarrow \alpha = 2,\ \beta = 6,\ \gamma = 12 \Rightarrow \alpha\beta\gamma = 144$$